Given: Sep 29. Due: Thursday, Oct 13 at the beginning of class

Homework Policy: You can consult class notes and books. Always try to solve the problems yourself; if you cannot make progress after some effort, you can discuss with your classmates or ask the instructor. However, you cannot copy other’s work: what you turn in must be your own. Make sure you are clear about the process you use to solve the problems: partial credit will be awarded.

Reading: Phillips Chapter 3, 4

Problem 1  Mass-Luminosity Relations: Theoretical

The gas pressure $P_g$ dominates over radiation pressure $P_r$ for stars with $M < 30M_\odot$. Free-free Kramer’s opacity $\kappa_{ff}$ dominates over electron scattering $\kappa_{es}$ when $M < 5M_\odot$.

Start with the ideal gas law and the definition of radiation pressure. Use hydrostatic equilibrium and the equation for radiative energy transport. Show that luminosity $L$ and mass $M$ have the following approximate scaling relations:

a. For low-mass stars with $< 5M_\odot$ (so free-free opacity and gas pressure dominate), show $L \propto M^5$ (this ignores convection, which can be important in low-mass stars).

b. For intermediate masses $5 – 30M_\odot$ (so gas pressure and electron scattering dominate), show $L \propto M^3$.

c. For high masses $\gg 30M_\odot$ (radiation pressure and electron scattering), show $L \propto M$

Problem 2  Neutron Star Crust

The outer layer of a neutron star consists of $^{56}$Fe ions embedded in a sea of electrons. [It is convenient to remember that 1 eV = $1.16 \times 10^4$ K, $\hbar c = 197.3$ MeV fm.]

a. What is the electron fraction $Y_e \equiv n_e/n_b$, where $n_b$ is the baryon number density

$n_b = \rho/m_p$? [Baryons are protons and neutrons, and you can ignore the difference between proton and neutron masses].
b. Show that the electron Fermi momentum \( p_F \) is given by:

\[
p_Fc = 1.11(Y_e\rho_{10})^{1/3}\text{MeV}
\]

where \( \rho_{10} = \rho/(10^{10}\text{kg m}^{-3}) \). At what density \( \rho = \rho_r \) do the electrons start to become relativistic?

c. Sometimes it is convenient to define a “Fermi Temperature”, \( T_F = E_F/k_B \). Physically, \( T_F \) is the temperature below which electrons are degenerate. Plot \( T_F \) as a function of \( \rho \) for the density range \( 10^3 - 10^{10}\text{kg m}^{-3} \).

d. What is the Fermi energy for the ions as a function of density, over the same range as part (c)? Think about this a bit: what type of particle are the ions, bosons or fermions? What statistics do they obey?

### Problem 3 Electron-Positron Creation

At high temperatures, photons can convert to electron-positron pairs and an equilibrium is established:

\[
\gamma + \gamma \leftrightarrow e^+ + e^-
\]

Recall that photons always have 0 chemical potential.

a. For \( T \ll m_e c^2/k_B \), we can treat the electrons and positrons as non-relativistic particles. Since the pair density is rather low, the pairs are very non-degenerate. Show that in equilibrium:

\[
n_n = n_+ = 4\left(\frac{m_e k_B T}{2\pi \hbar^2}\right)^3 e^{-2m_e c^2/k_B T}
\]

where \( n_+ \) and \( n_- \) are the number densities of positrons and electrons, respectively.

b. In general, \( n_- \neq n_+ \) because the medium may also contain ions, and charge neutrality requires \( n_- = n_+ + Z n_i \) (\( Z \) is the ion charge and \( n_i \) the ion number density). As \( T \) increases, the density of electrons from pair production becomes much larger than the density of electrons associated with ions, and then we have \( n_- = n_+ \) to a good approximation. Argue that in this case \( \mu(e^+) = \mu(e^-) = 0 \).

c. When \( T \gg m_e c^2/k_B \approx 6 \times 10^9\text{K} \) the electrons and positrons are extremely relativistic. Show that

\[
n_+ = n_- = \frac{1}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty dx \frac{x^2}{e^x + 1}
\]

To do this, go back to Eqn. 2.18 for the number of particles in a gas with energy \( \epsilon_p \) and temperature \( T \). Use this to get an expression for the number density, and then assume \( k_B T \gg m_e c^2 \) and that electrons/positrons are relativistic so \( \epsilon_p = pc \).

Compare this to the number density of photons, \( n_\gamma \). Which is larger, \( n_\gamma \) or \( n_- \)? [HINT: you do not need to evaluate the integral to know the answer.]
Problem 4  Phillips 3.3

Problem 5  Phillips 4.1

Problem 6  GS: Monte-Carlo Integration

A powerful technique for integrating problematic functions is known as Monte-Carlo integration. First developed during the Manhattan project, it uses random numbers to integrate functions. The simplest way is:

- Given an integral:
  \[ I = \int_{x_{\text{min}}}^{x_{\text{max}}} dx f(x) \]
- We sample \( N \) uniform random numbers over the interval \([x_{\text{min}}, x_{\text{max}}]\), \( x_i \)
- We then approximate the integral by:
  \[ I \approx \frac{x_{\text{max}} - x_{\text{min}}}{N} \sum_{i=1}^{N} f(x_i) \]

There are fancier ways to do this, as you can see at: [http://en.wikipedia.org/wiki/Monte_Carlo_integration](http://en.wikipedia.org/wiki/Monte_Carlo_integration).

We will do Monte-Carlo integration on the integral from Problem 3c:

\[ I = \int_{0}^{\infty} dx \frac{x^2}{e^x + 1} \]

Now, we know analytically that this is \( 3\zeta(3)/2 \approx 1.803085354 \), where \( \zeta(x) \) is the Riemann zeta function. But let’s try to do this numerically.

One problem here is that the limits of integration are infinite. But we can’t pick \( x_{\text{max}} = \infty \). So we need to try something else. If you plot the integrand \( f(x) = x^2/(e^x + 1) \) you will see that it peaks near \( x = 2 \) and decreases steadily after that. So some value of \( x_{\text{max}} \) that is > 2 will work. Another question is what value of \( N \) is best.

a. Write a routine that will do a Monte-Carlo integration of the integrand above for a given value of \( N \) and \( x_{\text{max}} \).

b. You should run this \( M = 100 \) times and determine how well your integration does. i.e., compare your results to the exact result given above. The most useful quantity to compute is:

\[ e(N, x_{\text{max}}) = \sqrt{\frac{1}{M} \sum_{j=1}^{M} \left( \frac{I_j - \frac{3}{2}\zeta(3)}{\frac{3}{2}\zeta(3)} \right)^2} \]

This is the root-mean-square (rms) fractional error.
c. Repeat this for a range of $x_{\text{max}}$ and $N$. How does $e(N, x_{\text{max}})$ depend on $N$? How does it depend on $x_{\text{max}}$ (plot them!). Does that make sense? What happens to your results as you increase $x_{\text{max}}$ and why? This problem led to solutions such as the Metropolis-Hastings algorithm.