Effect of Global Expansion on Bending of Null Geodesics

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Outline

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3. Null Geodesics in McVittie Spacetime
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The effect of the cosmological expansion on gravitational lensing has been studied for 30 years now. The issue has received an increased amount of attention in the past 10–15 years with observations leading to the conclusion that the global rate of expansion is accelerating. While the physical reason for the acceleration is not yet well understood, one possibility is a non-vanishing cosmological constant $\Lambda$. 

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The results in the literature so far on gravitational lensing and the cosmological expansion have been obtained using models with a constant value for $H(t) = H_0$, which gives the Kottler or Schwarzschild-de Sitter metric

$$d s^2 = - \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) d t^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2.$$
The McVittie Metric

- This metric was first discovered by G.C. McVittie in 1933, and can be thought of as representing a Schwarzschild black hole embedded in a FLRW spacetime. The McVittie line element in diagonal form is given by

\[
d s^2 = -\left(\frac{1 - \mu}{1 + \mu}\right)^2 d t^2 + (1 + \mu)^4 a^2(t) \, d \vec{x}^2,
\]

where \( a(t) \) is the asymptotic spatial scale factor and \( \mu := \frac{m}{2a(t)|\vec{x}|} \), with \( m \) being the mass of the central object.
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- One may introduce a coordinate transformation

\[ \vec{r} = (1 + \mu)^2 a(t) \, \vec{x} , \]

such that the geometrical meaning of the new radial coordinate \( r \) is tied to the area of the corresponding 2-sphere of constant \( r \).
The McVittie Metric

- Written in these co-moving coordinates, it has the form

\[ ds^2 = -(f - H^2 r^2) dt^2 - \frac{2Hr}{\sqrt{f}} dr dt + \frac{dr^2}{f} + r^2 d\Omega, \]

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- It is worth noting that the Schwarzschild-de Sitter or Kottler solution is a special case of the McVittie metric where \( H(t)^2 = \Lambda/3 \) is a constant.
Null Rays in Kottler Spacetime
(McVittie with constant $H_0$)

- The null line condition in Kottler spacetime is

$$-(f - H^2 r^2) (u^t)^2 - \frac{2 H r}{\sqrt{f}} u^t u^r + \left(\frac{u^r}{f}\right)^2 + r^2 (u^\phi)^2 = 0.$$ 

- There are two KVF’s in Kottler spacetime - $\xi = \partial / \partial t$ and $\eta = \partial / \partial \phi$; thus we may define the conserved quantities:

$$e = - (g_{00} \xi^0 u^0 + g_{01} \xi^0 u^1) = (f - H^2 r^2) u^t + \frac{H r}{\sqrt{f}} u^r$$

$$\ell = g_{33} \eta^3 u^3 = r^2 u^\phi.$$
After inserting these conserved quantities into the null line element, we obtain ($b$ is the impact parameter)

$$\frac{1}{\ell^2} \left( \frac{d\ell}{d\lambda} \right)^2 + \frac{1}{r^2} (f - H^2 r^2) = \frac{1}{b^2} \quad \text{where} \quad \frac{1}{b} = \frac{E}{\ell}.$$
After inserting these conserved quantities into the null line element, we obtain ($b$ is the impact parameter)

$$\frac{1}{\ell^2} \left( \frac{dr}{d\lambda} \right)^2 + \frac{1}{r^2} (f - H^2 r^2) = \frac{1}{b^2} \quad \text{where} \quad \frac{1}{b} = \frac{E}{\ell}.$$  

In Schwarzschild spacetime ($H = 0$) the deflection angle is

$$\alpha = \frac{4M}{b} + \frac{15\pi}{4} \frac{M^2}{b^2}.$$  

(from Arakida & Kasai, PRD 2012)
Null Rays in Kottler Spacetime

In Kottler spacetime the null line equation is the same as in Schwarzschild, with an effective impact parameter $B$,

$$\frac{1}{B^2} = \frac{1}{b^2} + H_0^2.$$

(From Arakida & Kasai, PRD 2012)

So the deflection angle is

$$\alpha = \frac{4M}{B} + \frac{15\pi}{4} \frac{M^2}{B^2}.$$

This was first shown in 2007 by Rindler and Ishak.
It is tempting to think that this replacement of the impact parameter implies that the effect of the cosmological constant is measurable in the deflection of starlight;
CAUTION

- It is tempting to think that this replacement of the impact parameter implies that the effect of the cosmological constant is measurable in the deflection of starlight;
- however this is not true \( \text{Lake, 2013} \), one cannot separate the effects of \( b = E/L \) and the effects of \( \Lambda \neq 0 \) in a physical measurement of bending of null geodesics.
When $H(t)$ is time dependent; there is no time-like KVF, thus the geodesic equations must be solved numerically. The geodesics for $u^t$ and $u^r$ have the form

$$\frac{du^t}{d\lambda} + \Gamma_{tt}^t (u^t)^2 + 2 \Gamma_{tr}^t u^t u^r + \Gamma_{rr}^t (u^r)^2 + \Gamma_{\phi\phi}^t (u^\phi)^2 = 0,$$

$$\frac{du^r}{d\lambda} + \Gamma_{tt}^r (u^t)^2 + 2 \Gamma_{tr}^r u^t u^r + \Gamma_{rr}^r (u^r)^2 + \Gamma_{\phi\phi}^r (u^\phi)^2 = 0.$$
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These equations are then combined with the null line element $(g_{\mu\nu} u^\mu u^\nu = 0)$ and integrated numerically using the parameterization that

$$H(t) = H_0 + at$$

and starting the integration at the distance of closest approach $R_0 = 1$. 
In each simulation, we choose values for the spacetime parameters $m$, $H_0$, $a$, and the conserved quantity $\ell$, and initial conditions for the $u^t$ and $u^r$ equations.

In practice, we choose values for $r(0)$, $\phi(0)$ and $u^r(0)$, while $u^t(0)$ is determined by imposing the normalization condition at $t = 0$. 
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Just as in the Kottler case, spacetime is not asymptotically flat and thus one cannot simply measure the bending angle by letting $r \to \pm \infty$ as one does in the Schwarzschild case. But, unlike in the Kottler case, $H(t)$ is now a time-dependent function.
Therefore, the amount of bending of the incoming geodesic in general is not the same as the bending of outgoing geodesic, so one may not simply calculate $\Delta \phi = 2\psi$ by doubling the difference in the direction of the light ray as it crosses the $\phi = 0$ and $\phi = \pi/2$ lines, and we must now directly compare the directions of the light ray as it crosses both the $\phi = 0$ and the $\phi = \pi$ lines.
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However, we will still find it convenient to start the simulations at $\phi(0) = \pi/2$ and choose initial conditions that include $u^r(0) = 0$ at that point, because this will give us a convenient way of setting the value of the distance of closest approach, $r(0)$. For each calculation of the light bending angle, therefore, we will run two simulations starting at $\phi = \pi/2$, one to $\phi = \pi$ and the other (in the “negative time direction”) to $\phi = 0$. 
null geodesics in kottler spacetime
null geodesics in mcvittie spacetime
conclusions

numerical results – forward bending of null geodesics

mass $M = 0.001$, closest approach $r(0) = 1.0$, $u^r(0) = 0.0$
Numerical Results – Backward Bending of Null Geodesics

Bending for the Backward null Geodesic

mass $M = 0.001$, closest approach $r(0) = 1.0$, $u^r(0) = 0.0$
Numerical Results – Total Bending of Null Geodesics

mass $M = 0.001$, closest approach $r(0) = 1.0, u' (0) = 0.0$
Conclusions

Points we can make:

- The results for the general dependence of the “forward” and “backward” bending angles on a agrees with what we would expect.

- This is work in progress, especially concerning its relationship with what is observed. We need to determine what are useful sets of parameter values, and what constant-$H$ is the most useful comparison for each accelerating spacetime.