Initial boundary value problem of the Z4c formulation of General Relativity

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Formulations of Numerical Relativity

BSSN-puncture gauge:
- BHs as coordinate singularity,

Generalized Harmonic gauge:
- Excision BH boundaries,

Is there another formulation with the strengths of both?
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- Sommerfeld BCs,
- No constraint damping scheme,
- Puncture gauge for advecting BHS.

**Generalized Harmonic gauge:**
- Excision BH boundaries,
- Trivial wave-like constraint subsystem,
- Well-posed IBVP (continuous dependence of ID even with BCs),
- Constraint preserving BCs,
- Constraint damping scheme,
- Dynamics control of coordinates through gauge sources

Is there another formulation with the strengths of both?
The Z4c formulation

A natural choice seems to be a conformal decomposition of Z4 (Bona *et. al.* 04-05, Bernuzzi-Hilditch 09, Alic *et. al.* 11)

\[ R_{\mu\nu} + \nabla_\mu Z_\nu + \nabla_\nu Z_\mu = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \]

\[ \Downarrow \]

\[ \tilde{\gamma}^{ij} = \gamma^{-\frac{1}{3}} \gamma^{ij}, \quad \hat{K} = \gamma^{ij} K_{ij} - 2\Theta, \quad \chi = \gamma^{-\frac{1}{3}}, \]

\[ \tilde{A}_{ij} = \gamma^{-\frac{1}{3}} (K_{ij} - \frac{1}{3} \gamma^{ij} K), \quad \tilde{\Gamma}^i = 2\tilde{\gamma}^{ij} Z_j + \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \tilde{\gamma}_{jk,l}. \]

- Conformal variables allow puncture evolution,
- Trivial wave-like constraint subsystem,
- Strongly hyperbolic with puncture gauge,
- Constraint damping.
Why boundary conditions?

- **Simple model**: evolution of a TOV star with BC at $r = 20M$

![Graph showing ρ(t)/ρ_c(0) over time](image)

**Sommerfeld case**: non-convergent reflections from the boundary affect the dynamics of the star.
Why boundary conditions?

- **Simple model**: evolution of a TOV star with BC at $r = 20M$

- **CP boundaries**: the absorbing properties of the BCs completely solve this problem.
Boundary conditions

How to impose BCs?

- Find out the in/outgoing modes,
- Specify BCs only on the incoming modes,

We want to impose BCs such that:

- The IBVP must be well-posed (continuous dependence of initial data),
- Constraint preservation (physical solution),
- Control of the incoming gravitational radiation.
High order BCs for Z4c

In 3D simulations we need to specify 10 BCs.

BCs for constraint subsector:

- The conditions which preserve the constraints are (4 conditions):

  \[(l^a \partial_a)^L \Theta \doteq 0,\]
  \[(l^a \partial_a)^L Z^i \doteq 0.\]

In flat space: \[(l^a \partial_a) = \partial_t - \partial_x.\]

* Why high? better absorption properties: \(L = 1, 2, 3, \ldots\)
In 3D simulations we need to specify 10 BCs.

**BCs physical subsector:**

- **Incoming-radiation-controlling condition** (2 conditions)
  \[
  (l^a \partial_a)^{L-1} \Psi_0 \doteq h \Psi_0 ,
  \]
  where the Newman-Penrose scalar \( \Psi_0 \) is defined by
  \[
  \Psi_0 = C_{\alpha\beta\nu\mu} l^\alpha m^\beta l^\nu m^\mu .
  \]
High order BCs for Z4c

Boundary conditions gauge subsector (4 conditions):

- **Lapse:**
  \[
  (i^{a} \partial_{a})^{L} (\partial_{t} - \beta \partial_{x}) \alpha \triangleq h_{\alpha} \quad i^{a} = \frac{1}{\sqrt{2}} (n^{a} + \sqrt{\mu_{L}} s^{a})
  \]

- **Longitudinal shift component:**
  \[
  (j^{a} \partial_{a})^{L} \left( \partial_{0} \Lambda + 4 \frac{\tilde{\mu} S - 1}{\tilde{\mu} S_{L} - 1} \Theta \right) \triangleq h_{\Lambda}
  \]

- **Transverse shift components:**
  \[
  \gamma^{i k} s_{[k q j]}^{A} \left( k^{a} \partial_{a} \right)^{L} \partial_{i} \beta_{j} \triangleq h^{A}
  \]
Kreiss-Agranovich-Métiever theory

- Consider an IBVP for a **first order strongly hyperbolic** PDE system

- Solve the boundary problem using the Laplace-Fourier transformation

\[ u(t, x, x^A) = \tilde{u}(x) \exp(st + i\omega_A x^A). \]

**Definition**

(Kreiss 70’s:) The above IBVP is called **boundary stable** if for all \( \text{Re}(s) > 0 \) and \( \omega \in \mathbb{R} \) there is a constant \( C \) such that

\[ |\tilde{u}(s, 0, \omega)| \leq C |\tilde{g}(s, \omega)|. \]

There is a symmetrizer \( H = H(s, \omega) \) which allows to show that the IBVP is well-posed via a standard **energy estimation** in the frequency domain.
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**IBVP for the Z4c system**

**IBVP for the Z4c system: \(L_2\)-norm estimate**

\[
\eta \| \alpha \|_{\eta,L+1,\Omega}^2 + \eta \sum_i \| \beta^i \|_{\eta,L+1,\Omega}^2 + \eta \sum_{ij} \| \gamma_{ij} \|_{\eta,L+1,\Omega}^2 + \\
\eta \| \alpha \|_{\eta,L+1,T}^2 + \eta \sum_i \| \beta^i \|_{\eta,L+1,T}^2 + \eta \sum_{ij} \| \gamma_{ij} \|_{\eta,L+1,T}^2 + \\
\leq C_{L+1} \left( \| h_\alpha \|_{\eta,L+1,T}^2 + \cdots + \| h_{\psi_0} \|_{\eta,L+1,T}^2 \right),
\]

The resulting IBVP for the Z4c formulation with the previous BCs is well-posed \(\rightarrow\) The solution does not grow arbitrarily fast.
Final comments

- We have specified high order boundary conditions for the gauge, constraint violating and the physical degrees of freedom,

- Using Kreiss-Agranovich-Métiever theory we have shown that the resulting IBVP is well-posed in the frozen coefficient approximation. It is expected that these results can be extended to the nonlinear case,

- The Z4c formulation has the strengths of both BSSN and GHG formulations,

- Some of these conditions have been implemented and tested numerically.

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Additional notes
Energy estimation

If the system is strongly hyperbolic we can define a Hermitian matrix such that

$$HP - P^T H = 0.$$  

The importance of the symmetrizer $H$ is related to the fact that

$$\langle u, \nu \rangle \equiv u^\dagger H \nu, \quad \Rightarrow \quad \|u\|^2 \equiv \langle u, u \rangle = u^\dagger H u.$$  

The norm defined above is usually called an energy norm.

- wave equation

$$\|u\|^2 = (\partial_t)^2 \phi + \nu^2 \sum_i (\partial_i \phi)^2.$$  

- Check the evolution of this norm.
Kreiss Theory

If the PDE system is strictly hyperbolic then (Kreiss 70’s)

**Definition**

If the above IBVP is boundary stable then it is strongly well-posed in the generalized sense. The solution \( u = u(t, x^i) \) satisfies the estimation

\[
\int_0^t \| u(\cdot, \tau) \|_\Sigma^2 \, d\tau + \int_0^t \| u(\cdot, \tau) \|_{\partial \Sigma}^2 \, d\tau \\
\leq K_T \left\{ \int_0^t \| F(\cdot, \tau) \|_\Sigma^2 \, d\tau + \int_0^t \| g(\cdot, \tau) \|_{\partial \Sigma}^2 \, d\tau \right\} ,
\]

in the interval \( 0 \leq t \leq T \) for a positive constant \( K_T \) which does not depend on \( F \) and \( g \). Here \( \| \cdot \|_\Sigma, \| \cdot \|_{\partial \Sigma} \) denote the \( L_2 \) norm with respect to the half-space and the boundary surface, respectively.

\[
\int_0^t \| u(\cdot, \tau) \|_\Sigma^2 \, d\tau + \int_0^t \| u(\cdot, \tau) \|_{\partial \Sigma}^2 \, d\tau \\
\leq K_T \left\{ \int_0^t \| F(\cdot, \tau) \|_\Sigma^2 \, d\tau + \int_0^t \| g(\cdot, \tau) \|_{\partial \Sigma}^2 \, d\tau \right\} ,
\]
Assume that the system is strongly hyperbolic and the eigenvalues of the principal symbol are

- real and pairwise distinct (strictly hyperbolic),
- real and have Z4 constant multiplicity \( \rightarrow \) Z4

if additionally the system is boundary stable

\[ |\tilde{u}(s, 0, \omega)| \leq C |\tilde{g}(s, \omega)| \]

then, there is a smooth symmetrizer \( H = H(s, \omega) \). Well-posedness is shown through an usual energy estimation.