Binary inspiral & post-Newtonian parameters from EMRI approximation in a radiation gauge

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I. Intro: Extreme-Mass-Ratio Inspiral

A stellar-size black hole orbiting a galactic black hole has mass ratio extreme enough

$$\frac{m}{M} \sim 10^{-3} \text{ to } 10^{-8}$$

that the waveform and trajectory can be found by modeling $m$ as a point particle.
At 0\textsuperscript{th} order in $m$ the particle moves on a geodesic of a background Kerr geometry.

At 1\textsuperscript{st} order it sees the perturbed metric, $h_{\alpha\beta}$.
The acceleration of a finite-mass particle relative to the background metric is called the *self-force* (per unit mass):

\[ f^\alpha := u^\beta \nabla_\beta u^\alpha = -\delta \Gamma^{\text{renormalized}}_{\beta\gamma} u^\beta u^\gamma \]
The divergent part of the perturbed metric gives the Coulomb part of the expression for the self force. It points radially inward and averages to zero.

Stronger statement (Gralla):
The renormalized self force in a regular gauge is

\[ f_{\mu}^{\text{renormalized}} = \lim_{\rho \to 0} \langle f_{\mu} \rangle \]
The self-force (departure from geodesic motion) to order $m/M$ has two parts:

Dissipative part associated with the loss of energy to gravitational waves,

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odd under ingoing ⇔ outgoing
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Conservative part

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even under ingoing ⇔ outgoing
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II. High-order post-Newtonian parameters describing binary inspiral

The post-Newtonian waveform of a (non-eccentric) inspiral can be found from $E$ and $dE/dt$

Parameters in the PN expansion of these expressions can be obtained, to first-order in the mass ratio from the EMRI approximation.

Blanchet, Detweiler, Le Tiec, Whiting
An (almost) gauge invariant quantity:

The perturbed spacetime of a particle in circular orbit still has the helical symmetry of the background Kerr or Schwarzschild geometry:

\[ u^\alpha = U k^\alpha, \]

\[ k^\alpha = (t^\alpha + \Omega \phi^\alpha) \]

\( \Delta U \)renormalized at constant angular velocity is invariant under gauge transformations generated by helically symmetric gauge vectors.
Post-Newtonian expansion of $\Delta U$

With $r$ defined by the Keplerian relation

$$r = M (M \Omega)^{-2/3}$$

and $M=1$, the expansion of the (renormalized) $\Delta U$ has the form

$$\Delta U = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \cdots$$

$$-\ln r \left[ \frac{\beta_4}{r^5} + \frac{\beta_5}{r^6} + \cdots \right]$$

$$+(\ln r)^2 \left[ \frac{\gamma_7}{r^8} + \cdots \right] + (\ln r)^3 \left[ \frac{\delta_{10}}{r^{11}} + \cdots \right] + \cdots$$
A surprise

The EMRI matching finds an unexpected $1/r^{6.5}$ (5.5 PN) coefficient.

This appears to be due, at least in part, to a “tail of tails” contribution (Blanchet).
Elaborate checks

- Independent calculation in a Regge-Wheeler gauge, with agreement to current numerical accuracy of 350 places.

- Compared the EMRI energy loss $\frac{dE}{dt}/E$ with the high-order PN expression for the flux at $I^+$ (Fujita) and future horizon (Tagoshi, Mano, Tagasuki).

- Compared value of $\Delta U$ with Ottewill and Casals, obtaining agreement to their numerical accuracy.
Finally, computing the PN EMRI series analytically to this order, using a formalism by Mano, Suzuki, Takasugi:

\[-\frac{1}{r} - \frac{2}{r^2} - \frac{5}{r^3} + \frac{-121}{3} \frac{\pi^2}{r^4} + \frac{41}{32} \frac{\pi^2}{r^4} + \frac{64 \log[r]}{5} - \frac{956 \log[r]}{105} - \frac{13696 \pi}{525 r^{13/2}}\]
The extreme precision of the numerical calculation + experience suggesting a subset of analytic coefficients are rational or rational x \pi allows one to numerically to find exact PN coefficients to very high pN order!

\[
\begin{align*}
\frac{1}{r} & - \frac{2}{r^2} - \frac{5}{r^3} + \frac{-121}{3} \frac{\pi^2}{r^4} + \frac{41}{32} \frac{\pi^2}{r^4} + \frac{64}{5} \frac{\log[r]}{r^5} - \frac{956}{105} \frac{\log[r]}{r^6} + \frac{-13696}{525} \frac{\pi}{r^{13/2}} + \\
& - \frac{51256}{567} \frac{\log[r]}{r^7} + \frac{81077}{3675} \frac{\pi}{r^{15/2}} + \frac{27392}{525} \frac{\log[r]^2}{r^8} + \frac{82561159}{467775} \frac{\pi}{r^{17/2}} + \frac{-27016}{2205} \frac{\log[r]^2}{r^9} + \\
& - \frac{11723776}{55125} \frac{\pi}{r^{19/2}} + \frac{-4027582708}{9823275} \frac{\log[r]^2}{r^{10}} + \frac{99186502}{1157625} \frac{\pi}{r^{21/2}} + \frac{23447552}{165375} \frac{\log[r]^3}{r^{11}}
\end{align*}
\]
Example: Numerically find

\[ \beta_6 = -90.398589065255731922 \quad 398589065255731922 \]

\[ 398589065255731922 \quad 398589065255731922 \]

\[ 398589065255731922 \quad 398589065255731922 \]

\[ 398589048525 \]

\[ \beta_{6,\text{numerical}} \quad \frac{51256}{567} = -6.4 \times 10^{-113} \]

Assuming the exact analytic expression has numerator and denominator each with less than 10 digits and power of \( \pi \) is 1, this is the unique matching rational.

If assume only that the power of \( \pi \) is an integer < 10, probability of an accidental match is < \( 10^{-90} \).
Other coefficients involve combinations of other irrational numbers and we find only their numerical values:

\[
\Delta U = -\frac{1}{r} + \sum_{n=1}^{23} \alpha_n \frac{1}{r^{n+1}} + \sum_{n=4}^{10} \beta_n \frac{\log r}{r^{n+1}} + \sum_{n=7}^{10} \gamma_n \frac{\log^2 r}{r^{n+1}} + \sum_{n=10} \delta_n \frac{\log^3 r}{r^{n+1}} + \ldots
\]

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<th>Coefficient</th>
<th>Numerical value</th>
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Spin-dependent parameters from Kerr: preliminary results

First EMRI computations of $\Delta U$ completed last year (Shah, Keidl, JF ‘12; Dolan, Barack, Wardell)

Higher accuracy computation begun for small values of $a/M$ to obtain spin-orbit and spin-spin terms in PN expansion. Shah, Le Tiec, JF
\[ \Delta U - \Delta U(a = 0) = \frac{a}{M} \left[ \frac{A}{r^{5/2}} + \frac{B}{r^{7/2}} + \frac{C}{r^{9/2}} + \cdots \right] \]

\[ [\Delta U - \Delta U(a = 0)] \times r^{5/2} \]

Le Tiec, Shah, JF
\[ \Delta U - \Delta U(a = 0) = \frac{a}{M} \left[ \frac{A}{r^{5/2}} + \frac{B}{r^{7/2}} + \frac{C}{r^{9/2}} + \cdots \right] \]

\[ \left[ \Delta U - \Delta U(a = 0) \right] \times r^{5/2} \]

Le Tiec, Shah, JF
Comparisons

Sago Barack Detweiler 08, Shah et al 10
Le Tiec, Mroue, Barack, Buonanno, Pfeiffer, Sago, Taracchini 11 (NR)
Blanchet Detweiler Le Tiec Whiting (PN) 10
Barack Damour Sago 10 (PN)

Unreasonable effectiveness if use symmetric mass ratio

\[ \nu = \frac{m_1 m_2}{(m_1 + m_2)^2} \]
Weyl scalar $\psi_0$

Hertz potential $\Psi$

metric perturbation $h_{\alpha\beta}$

$f^\alpha$

$f_{\alpha, \text{renormalized}}$
Comparisons

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Le Tiec, Mroue, Barack, Buonanno, Pfeiffer, Sago, Taracchini 11 (NR)

Blanchet Detweiler Le Tiec Whiting (PN) 10
Barack Damour Sago 10 (PN)
$h_{\alpha\beta}$ approaches the linearized Schwarzschild metric:

$$h_{\alpha\beta} = \frac{2m}{\rho} \left( g_{\alpha\beta} + 2u_\alpha u_\beta \right) + O(\rho^0)$$

In Newtonian language, this gives the Coulomb part of the self-force. It points radially inward, averages to zero and does not contribute to the self-force.
Mode-sum renormalization (Barack-Ori, . . . )

In mode-sum renormalization, one subtracts the Coulomb contribution, essentially

\[ \nabla \frac{1}{\rho} \]

writing \( \nabla \frac{1}{\rho} \) as a sum of spherical harmonics with respect to background spherical coordinates.
EMRI computation accurate only to first order in mass ratio
\[ \frac{m_1 m_2}{(m_1 + m_2)^2} \]
but to all orders in PN.
Use numerical self-force results for $\Delta U$ to obtain PN parameters.

First computations
Blanchet, Detweiler, Le Tiec, Whiting ‘10