Tensors in Mathematica 9: Built-In Capabilities

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This Talk

I intend to cover four main topics:

- How to make tensors in the newest version of Mathematica.
- The metric tensor and how to transform vectors into covectors.
- Cartesian tensor operations.
- GR Operations
How to Build a Tensor in Mathematica 9

Rank One

For rank one tensors, we can write them as tangent vectors,

\[
\begin{array}{c}
\text{Table}[\{x^i\}, \{i, \{"1", "2", "3"\}\}] // \text{MatrixForm}\\
\begin{pmatrix}
x^1 \\
x^2 \\
x^3 
\end{pmatrix}
\]

or as covectors

\[
\text{Table}[x_i, \{i, \{"1", "2", "3"\}\}]
\]

\{x_1, x_2, x_3\}
Rank Two and Higher

We can use similar methods to develop rank two tensors, though *Mathematica* is not able to cope with abstract indices without help from third-party software—I like xAct.

Table\[\sigma^{ij}, \{i, \{x, y, z\}\}, \{j, \{x, y, z\}\}\] // MatrixForm

\[
\begin{pmatrix}
\sigma^{xx} & \sigma^{xy} & \sigma^{xz} \\
\sigma^{yx} & \sigma^{yy} & \sigma^{yz} \\
\sigma^{zx} & \sigma^{zy} & \sigma^{zz}
\end{pmatrix}
\]

Table\[\sigma^{ijkl}, \{i, \{x, y, z\}\}, \{j, \{x, y, z\}\}, \{k, \{x, y, z\}\}\] // MatrixForm

\[
\begin{pmatrix}
\sigma^{xx} & \sigma^{xy} & \sigma^{xz} \\
\sigma^{yx} & \sigma^{yy} & \sigma^{yz} \\
\sigma^{zx} & \sigma^{zy} & \sigma^{zz}
\end{pmatrix}
\]

You can produce the individual tensor components,

\[
\sigma[i\_j\_n\_] := 
\]

Table\[\left(\eta \left(\partial_{x_1} v_1[x_j] + \partial_{x_2} v_2[x_j] - \text{If}[i = j, \frac{2}{3} \partial_{x_3} v_3[x_k], 0] + \text{If}[i = j, \xi \partial_{x_3} v_3[x_k], 0]\right) + \text{If}[i = j, \xi \partial_{x_3} v_3[x_k], 0]\right), \\
\{k, 1, n, 1\}\] // MatrixForm
\{\sigma[1, 1, 3], \sigma[1, 2, 3], \sigma[1, 3, 3]\} // TraditionalForm

\[
\begin{align*}
\frac{4}{3} \eta v_1'(x_1) + \xi v_1'(x_1) \\
\eta \left(2 v_1'(x_1) - \frac{2}{3} v_2'(x_2) + \xi v_2'(x_2)\right) \\
\eta \left(2 v_1'(x_1) - \frac{2}{3} v_3'(x_3) + \xi v_3'(x_3)\right)
\end{align*}
\]

\[
\begin{align*}
\eta \left(v_1'(x_2) + v_2'(x_2)\right) \\
\frac{4}{3} \eta v_2'(x_2) + \xi v_2'(x_2) \\
\eta \left(2 v_2'(x_2) - \frac{2}{3} v_3'(x_3) + \xi v_3'(x_3)\right)
\end{align*}
\]

\[
\begin{align*}
\eta \left(v_1'(x_3) + v_3'(x_3)\right) \\
\frac{4}{3} \eta v_3'(x_3) + \xi v_3'(x_3) \\
\eta \left(2 v_3'(x_3) - \frac{2}{3} v_1'(x_1) + \xi v_1'(x_1)\right)
\end{align*}
\]

You can also write a table to produce the entire tensor

\[
st[n_\_] := \text{Table}[\sigma[i, j, n], \{i, 1, n, 1\}, \{j, 1, n, 1\}] // MatrixForm
\]

\[
st[3] // TraditionalForm
\]

\[
\begin{align*}
\frac{4}{3} \eta v_1'(x_1) + \xi v_1'(x_1) \\
\eta \left(2 v_1'(x_1) - \frac{2}{3} v_2'(x_2) + \xi v_2'(x_2)\right) \\
\eta \left(2 v_1'(x_1) - \frac{2}{3} v_3'(x_3) + \xi v_3'(x_3)\right)
\end{align*}
\]

\[
\begin{align*}
\eta \left(v_1'(x_2) + v_2'(x_2)\right) \\
\frac{4}{3} \eta v_2'(x_2) + \xi v_2'(x_2) \\
\eta \left(2 v_2'(x_2) - \frac{2}{3} v_3'(x_3) + \xi v_3'(x_3)\right)
\end{align*}
\]

\[
\begin{align*}
\eta \left(v_1'(x_3) + v_3'(x_3)\right) \\
\frac{4}{3} \eta v_3'(x_3) + \xi v_3'(x_3) \\
\eta \left(2 v_3'(x_3) - \frac{2}{3} v_1'(x_1) + \xi v_1'(x_1)\right)
\end{align*}
\]
The Metric Tensor

A specific example of a calculation is one where we transform from a tangent vector to a covector using the metric tensor,

\[ v_i = g_{ij} v^j \]  

(1)

So, given the tangent vector \( v^j = (2, r \cos \theta, -r \phi) \), and assuming we are in spherical coordinates, we can find the metric.

```plaintext
tv = {2, r Cos[\[Theta]], -r \[Phi]};
met = CoordinateChartData["Spherical", "Metric", {r, \[Theta], \[Phi]}] // TraditionalForm
```

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2(\theta)
\end{pmatrix}
\]

We can even find the inverse metric,

```plaintext
im = CoordinateChartData["Spherical", "InverseMetric", {r, \[Theta], \[Phi]}] // TraditionalForm
```

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 \\
0 & 0 & \frac{\csc^2(\theta)}{r^2}
\end{pmatrix}
\]
We take the product of the inverse metric with the tangent vector,

\[ tv \cdot \text{CoordinateChartData}["Spherical", "InverseMetric", \{r, \theta, \phi\}] \]

\[ \text{TraditionalForm} \]

giving us the covector

\[ \left\{ 2, \frac{\cos(\theta)}{r}, -\frac{\phi \csc^2(\theta)}{r} \right\} \]
The Cartesian Tensor

We create a new tensor,

\[
\text{newtensor} = \text{Table}[x_{\text{Grid}[\{1\}]} y_{\text{Grid}[\{1\}]} z_{\text{Grid}[\{3\}]}, \{i, 1, 3\}, \{j, 1, 3\}, \{k, 1, 3\}];
\]

\[
\text{newtensor} // \text{MatrixForm} // \text{TraditionalForm}
\]

We can see if its a tensor,

\[
\text{ArrayQ[newtensor]}
\]

True

We can find its rank

\[
\text{TensorRank[newtensor]}
\]

3
We can perform a contraction

$$\text{contens} = \text{Table} \left[ \sum_{z=1}^{3} \text{newtensor}[[i, a, a], [i, 1, 3]] \right] \text{// MatrixForm} \text{// TraditionalForm}$$

$$\left( x^1 y^1 z^1 + x^1 y^2 z^2 + x^1 y^3 z^3 \right)$$
$$\left( x^2 y^1 z^1 + x^2 y^2 z^2 + x^2 y^3 z^3 \right)$$
$$\left( x^3 y^1 z^1 + x^3 y^2 z^2 + x^3 y^3 z^3 \right)$$

We have the inner product

$$\text{inl} = \text{Inner}[\text{Times}, \text{newtensor}, \text{newtensor}, \text{Plus}] \text{// FullSimplify} \text{// MatrixForm}$$

$$\left( x^1 y^2 z^1 \left( x^1 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^1 y^1 z^2 \left( x^1 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^1 y^2 z^3 \left( x^1 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$

$$\left( x^2 y^1 z^1 \left( x^2 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^2 y^2 z^2 \left( x^2 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^2 y^3 z^3 \left( x^2 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$

$$\left( x^3 y^1 z^1 \left( x^3 z^1 + x^3 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^3 y^2 z^2 \left( x^3 z^1 + x^3 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^3 y^3 z^3 \left( x^3 z^1 + x^3 z^2 + x^3 z^3 \right) \right)$$

$$\text{newtensor} . \text{newtensor} \text{// FullSimplify} \text{// MatrixForm}$$

$$\left( x^1 y^1 z^1 \left( x^1 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^1 y^2 z^2 \left( x^1 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^1 y^3 z^3 \left( x^1 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$

$$\left( x^2 y^2 z^1 \left( x^2 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^2 y^1 z^2 \left( x^2 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^2 y^3 z^3 \left( x^2 z^1 + x^2 z^2 + x^3 z^3 \right) \right)$$

$$\left( x^3 y^2 z^2 \left( x^3 z^1 + x^3 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^3 y^1 z^3 \left( x^3 z^1 + x^3 z^2 + x^3 z^3 \right) \right)$$
$$\left( x^3 y^2 z^3 \left( x^3 z^1 + x^3 z^2 + x^3 z^3 \right) \right)$$
We can take the direct product,

\[
\text{in1} = \text{Outer}[\text{Times}, \text{newtensor}, \text{newtensor}] // \text{FullSimplify} // \text{MatrixForm}
\]

A very large output was generated. Here is a sample of it:

\[
\begin{pmatrix}
  x^2 & y^2 & z^2 \\
  x^2 & y^2 & z^2 \\
  x^2 & y^2 & z^2 \\
\end{pmatrix}
\begin{pmatrix}
  x^2 & y^2 & z^2 \\
  x^2 & y^2 & z^2 \\
  x^2 & y^2 & z^2 \\
\end{pmatrix}
\begin{pmatrix}
  x^2 & y^2 & z^2 \\
  x^2 & y^2 & z^2 \\
  x^2 & y^2 & z^2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
\end{pmatrix}
\begin{pmatrix}
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
\end{pmatrix}
\begin{pmatrix}
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
  \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle & \langle \langle 1 \rangle \rangle \\
\end{pmatrix}
\]
We can take the trace of the tensor,
\[ \text{Tr}[\text{newtensor}] \]
\[ x^1 y^1 z^1 + x^2 y^2 z^2 + x^3 y^3 z^3 \]

We can transpose,
\[ \text{Transpose}[\text{newtensor}] \text{ // MatrixForm // TraditionalForm} \]
\[
\begin{pmatrix}
  (x^1 y^1 z^1) & (x^2 y^1 z^1) & (x^3 y^1 z^1) \\
  (x^1 y^1 z^2) & (x^2 y^1 z^2) & (x^3 y^1 z^2) \\
  (x^1 y^1 z^3) & (x^2 y^1 z^3) & (x^3 y^1 z^3)
\end{pmatrix}
\]

We can take a partial derivative
\[ \partial_x \text{newtensor} \text{ // MatrixForm // TraditionalForm} \]
\[
\begin{pmatrix}
  (1 x^{1-1} y^1 z^1) & (1 x^{1-1} y^2 z^1) & (1 x^{1-1} y^3 z^1) \\
  (1 x^{1-1} y^1 z^2) & (1 x^{1-1} y^2 z^2) & (1 x^{1-1} y^3 z^2) \\
  (1 x^{1-1} y^1 z^3) & (1 x^{1-1} y^2 z^3) & (1 x^{1-1} y^3 z^3)
\end{pmatrix}
\]
\[
\begin{pmatrix}
  (2 x^{2-1} y^1 z^1) & (2 x^{2-1} y^2 z^1) & (2 x^{2-1} y^3 z^1) \\
  (2 x^{2-1} y^1 z^2) & (2 x^{2-1} y^2 z^2) & (2 x^{2-1} y^3 z^2) \\
  (2 x^{2-1} y^1 z^3) & (2 x^{2-1} y^2 z^3) & (2 x^{2-1} y^3 z^3)
\end{pmatrix}
\]
\[
\begin{pmatrix}
  (3 x^{3-1} y^1 z^1) & (3 x^{3-1} y^2 z^1) & (3 x^{3-1} y^3 z^1) \\
  (3 x^{3-1} y^1 z^2) & (3 x^{3-1} y^2 z^2) & (3 x^{3-1} y^3 z^2) \\
  (3 x^{3-1} y^1 z^3) & (3 x^{3-1} y^2 z^3) & (3 x^{3-1} y^3 z^3)
\end{pmatrix}
\]
we can determine the differentials

\[
\begin{align*}
\text{Dt[newtensor]} \ // \ MatrixForm \ // \ TraditionalForm
\end{align*}
\]

\[
\begin{pmatrix}
\frac{\partial}{\partial x^1} x^{1-1} y^1 z^1 + \frac{\partial}{\partial y} x^1 y^{1-1} z^1 + \frac{\partial}{\partial z} x^1 y^1 z^{1-1} \\
\frac{\partial}{\partial x^2} x^{2-1} y^1 z^1 + \frac{\partial}{\partial y} x^2 y^{1-1} z^1 + \frac{\partial}{\partial z} x^2 y^1 z^{1-1} \\
\frac{\partial}{\partial x^3} x^{3-1} y^1 z^1 + \frac{\partial}{\partial y} x^3 y^{1-1} z^1 + \frac{\partial}{\partial z} x^3 y^1 z^{1-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial}{\partial x^1} x^{1-1} y^2 z^1 + \frac{\partial}{\partial y} x^1 y^{2-1} z^1 + \frac{\partial}{\partial z} x^1 y^2 z^{1-1} \\
\frac{\partial}{\partial x^2} x^{2-1} y^2 z^1 + \frac{\partial}{\partial y} x^2 y^{2-1} z^1 + \frac{\partial}{\partial z} x^2 y^2 z^{1-1} \\
\frac{\partial}{\partial x^3} x^{3-1} y^2 z^1 + \frac{\partial}{\partial y} x^3 y^{2-1} z^1 + \frac{\partial}{\partial z} x^3 y^2 z^{1-1}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial}{\partial x^1} x^1 y^{1-2} z^2 + \frac{\partial}{\partial y} x^1 y^{1-1} z^2 + \frac{\partial}{\partial z} x^1 y^1 z^1 \\
\frac{\partial}{\partial x^2} x^2 y^{1-2} z^2 + \frac{\partial}{\partial y} x^2 y^{1-1} z^2 + \frac{\partial}{\partial z} x^2 y^1 z^1 \\
\frac{\partial}{\partial x^3} x^3 y^{1-2} z^2 + \frac{\partial}{\partial y} x^3 y^{1-1} z^2 + \frac{\partial}{\partial z} x^3 y^1 z^1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial}{\partial x^1} x^{1-2} y^2 z^3 + \frac{\partial}{\partial y} x^{1-1} y^2 z^3 + \frac{\partial}{\partial z} x^1 y^2 z^1 \\
\frac{\partial}{\partial x^2} x^{2-2} y^2 z^3 + \frac{\partial}{\partial y} x^{2-1} y^2 z^3 + \frac{\partial}{\partial z} x^2 y^2 z^1 \\
\frac{\partial}{\partial x^3} x^{3-2} y^2 z^3 + \frac{\partial}{\partial y} x^{3-1} y^2 z^3 + \frac{\partial}{\partial z} x^3 y^2 z^1
\end{pmatrix}
\]
We can look for symmetries,

\texttt{TensorSymmetry[newtensor]} \\
\{\}

\texttt{symten = Symmetrize[newtensor, Antisymmetric[{1, 2}]]}

\texttt{StructuredArray[SymmetrizedArray, \{3, 3\}, \{-\text{Structured Data}\} - \text{Normal}@symten]} \\
\texttt{Normal[symten] // MatrixForm}

\[
\begin{bmatrix}
\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2} (-x^2 y^1 z^1 + x^1 y^2 z^1) \\ \frac{1}{2} (-x^2 y^1 z^2 + x^1 y^2 z^2) \\ \frac{1}{2} (-x^2 y^1 z^3 + x^1 y^2 z^3) \end{bmatrix} & \begin{bmatrix} \frac{1}{2} (-x^1 y^1 z^1 + x^2 y^3 z^1) \\ \frac{1}{2} (-x^1 y^1 z^2 + x^2 y^3 z^2) \\ \frac{1}{2} (-x^1 y^1 z^3 + x^2 y^3 z^3) \end{bmatrix} \\
\begin{bmatrix} \frac{1}{2} (x^2 y^1 z^1 - x^1 y^2 z^1) \\ \frac{1}{2} (x^2 y^1 z^2 - x^1 y^2 z^2) \\ \frac{1}{2} (x^2 y^1 z^3 - x^1 y^2 z^3) \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2} (-x^1 y^2 z^1 + x^2 y^3 z^1) \\ \frac{1}{2} (-x^1 y^2 z^2 + x^2 y^3 z^2) \\ \frac{1}{2} (-x^1 y^2 z^3 + x^2 y^3 z^3) \end{bmatrix} \\
\begin{bmatrix} \frac{1}{2} (x^3 y^1 z^1 - x^1 y^3 z^1) \\ \frac{1}{2} (x^3 y^1 z^2 - x^1 y^3 z^2) \\ \frac{1}{2} (x^3 y^1 z^3 - x^1 y^3 z^3) \end{bmatrix} & \begin{bmatrix} \frac{1}{2} (x^1 y^2 z^1 - x^2 y^3 z^1) \\ \frac{1}{2} (x^1 y^2 z^2 - x^2 y^3 z^2) \\ \frac{1}{2} (x^1 y^2 z^3 - x^2 y^3 z^3) \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\end{bmatrix}
\]

\texttt{TensorSymmetry[symten]} \\
\texttt{Antisymmetric[{1, 2}]}
GR Operations

The Metric

Here we input the metric

```math
Metric[gi_] := Module[
{n = Length[gi], g, ing},
g = Table[If[μ ≥ ν, gi[[μ, ν]], gi[[ν, μ]]], {μ, n}, {ν, n}];
ing = Simplify[Inverse[g]];
{n, g, ing}]
```

we assign labels to the coordinates

```math
Evaluate[Table[x[μ], {μ, 4}]] = {t, r, θ, ψ};
```

Here is the Schwarzschild metric:

```math
Metric[{{1 - 1/r}, {0, -1/(1 - 1/r)}, {0, 0, -r^2}, {0, 0, 0, -r^2 2 Sin[θ]^2}}] //
TraditionalForm
```

```math
Out[43]//TraditionalForm =
```

```
\[
\begin{pmatrix}
1 - \frac{1}{r} & 0 & 0 & 0 \\
0 & -\frac{1}{1 - \frac{1}{r}} & 0 & 0 \\
0 & 0 & -r^2 & 0 \\
0 & 0 & 0 & -2r^2\sin^2(θ)
\end{pmatrix}
\]

```

```
\[
\begin{pmatrix}
\frac{r}{r-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{r^2} \\
0 & 0 & 0 & -\frac{\csc^2(θ)}{2r^2}
\end{pmatrix}
\]

```
Here we have the Christoffel Symbols:

\[ \nabla \text{Christoffel} \{ n_, g_, \text{ing_}, \text{OptionsPattern[]} \} := \text{Module} \left[ \right. \\
\left. \{ \Gamma, \text{in} \Gamma \}, \Gamma = \text{in} \Gamma = \text{Table}[0, \{ \lambda, n \}, \{ \mu, n \}, \{ \nu, n \}] \right. \\
\left. \quad \text{Do} \left[ \right. \\
\left. \quad \Gamma[\lambda, \mu, \nu] = \text{Simplify}[D[g[\lambda, \nu], x[\mu]] + D[g[\lambda, \mu], x[\nu]] - D[g[\mu, \nu], x[\lambda]] / 2]; \right. \\
\left. \quad \text{If}[\mu \neq \nu, \Gamma[\lambda, \nu, \mu] = \Gamma[\lambda, \mu, \nu]]; \right. \\
\left. \quad \text{Do}[\text{If}[\text{in} \Gamma[\lambda, \mu, \nu] = \text{Simplify}[\text{Sum}[\text{ing}[[\lambda, \rho]] \Gamma[[\rho, \mu, \nu]], \{\rho, n\}]]; \right. \\
\left. \quad \text{If}[\mu \neq \nu, \text{in} \Gamma[[\lambda, \nu, \mu]] = \text{in} \Gamma[[\lambda, \mu, \nu]], \{\lambda, n\}, \{\mu, n\}, \{\nu, n\}]; \right. \\
\left. \quad \text{If}[\text{OptionValue}[\text{PrintNonZero}], \right. \\
\left. \quad \text{Do}[\text{If}[\text{in} \Gamma[[\lambda, \mu, \nu]] = 0, \text{Print}["\text{in} \Gamma[[(\lambda-1, \mu-1, \nu-1)]] = 0", \Gamma[[\lambda, \mu, \nu]]]), \right. \\
\left. \quad \text{Do}[\text{If}[\text{in} \Gamma[[\lambda, \mu, \nu]] = 1, 0, \right. \\
\left. \quad \text{in} \Gamma[\lambda, \mu, \nu], \{\lambda, n\}, \{\mu, n\}, \{\nu, n\}]]. \right. \\
\left. \{ \Gamma, \text{in} \Gamma \} \right. \\
\right] \]

\[ \text{Christoffel} \{ n_, g_, \text{ing_}, \text{OptionsPattern[]} \} := \text{Module} \left[ \right. \\
\left. \{ \Gamma, \text{in} \Gamma \}, \Gamma = \text{in} \Gamma = \text{Table}[0, \{ \lambda, n \}, \{ \mu, n \}, \{ \nu, n \}] \right. \\
\left. \quad \text{Do} \left[ \right. \\
\left. \quad \Gamma[\lambda, \mu, \nu] = \text{Simplify}[D[g[\lambda, \nu], x[\mu]] + D[g[\lambda, \mu], x[\nu]] - D[g[\mu, \nu], x[\lambda]] / 2]; \right. \\
\left. \quad \text{If}[\mu \neq \nu, \Gamma[\lambda, \nu, \mu] = \Gamma[\lambda, \mu, \nu]]; \right. \\
\left. \quad \text{Do}[\text{If}[\text{in} \Gamma[\lambda, \mu, \nu] = \text{Simplify}[\text{Sum}[\text{ing}[[\lambda, \rho]] \Gamma[[\rho, \mu, \nu]], \{\rho, n\}]]; \right. \\
\left. \quad \text{If}[\mu \neq \nu, \text{in} \Gamma[[\lambda, \nu, \mu]] = \text{in} \Gamma[[\lambda, \mu, \nu]], \{\lambda, n\}, \{\mu, n\}, \{\nu, n\}]; \right. \\
\left. \quad \text{Do}[\text{If}[\text{in} \Gamma[[\lambda, \mu, \nu]] = 1, 0, \right. \\
\left. \quad \text{in} \Gamma[\lambda, \mu, \nu], \{\lambda, n\}, \{\mu, n\}, \{\nu, n\}]]. \right. \\
\left. \{ \Gamma, \text{in} \Gamma \} \right. \\
\right] \]

Here we calculate the Christoffel symbols for the Schwarzschild Metric

\[ \text{Christoffel} \left[ \text{Metric}[[1 - 1/r], \{0, -1/(1 - 1/r)\}, \{0, 0, -r^2\}, \{0, 0, 0, -r^2 \text{Sin[\theta]^2}\}] \right]; \]
\[
\begin{align*}
\Gamma^0_{0 0} & = \frac{1}{2} \frac{1}{r^2} \\
\Gamma^1_{0 0} & = -\frac{1}{2} \frac{1}{r^2} \\
\Gamma^1_{1 1} & = \frac{1}{2} \frac{1}{(-1 + r)^2} \\
\Gamma^2_{2 2} & = r \\
\Gamma^2_{3 3} & = r \sin^2(\theta) \\
\Gamma^2_{2 1} & = -r \\
\Gamma^2_{3 3} & = r^2 \cos(\theta) \sin(\theta) \\
\Gamma^3_{3 1} & = -r \sin^2(\theta) \\
\Gamma^3_{3 2} & = -r^2 \cos(\theta) \sin(\theta) \\
\Gamma^0_{1 0} & = \frac{1}{2} \frac{1}{(-1 + r) r} \\
\Gamma^1_{0 0} & = \frac{-1 + r}{2 r^2} \\
\Gamma^1_{1 1} & = \frac{1}{2} \frac{1}{r - 2 r^2} \\
\Gamma^2_{2 2} & = 1 - r \\
\Gamma^2_{3 3} & = (-1 + r) \sin^2(\theta) \\
\Gamma^2_{2 1} & = \frac{1}{r} \\
\Gamma^3_{3 3} & = -\cos(\theta) \sin(\theta) \\
\Gamma^3_{3 1} & = -\frac{1}{r} \\
\Gamma^3_{3 2} & = \cot(\theta)
\end{align*}
\]
Here we calculate the components of the Riemann tensor:

```mathematica
Riemann[n_, g_, ing_] := Module{
    {Γ, inΓ, R = Table[0, {α, n}, {β, n}, {μ, n}, {ν, n}], R2 = Table[0, {μ, n}, {ν, n}], R0},
    {Γ, inΓ} = Christoffel[{n, g, ing}]; Do[R[[α, β, μ, ν]] = R[[β, α, ν, μ]] =
      Simplify[Sum[g[[α, λ]] (D[inΓ[[λ, β, ν]], x[μ]] - D[inΓ[[λ, β, μ]], x[ν]]) +
          Γ[[α, λ, μ]] inΓ[[λ, β, ν]] - Γ[[α, λ, ν]] inΓ[[λ, β, μ]], {λ, n}]];  
    R[[β, α, μ, ν]] = R[[α, β, ν, μ]] = -R[[α, β, μ, ν]];  
    If[μ ≠ α, R[[μ, ν, α, β]] = R[[ν, μ, β, α]] = R[[α, β, μ, ν]];  
    R[[ν, μ, α, β]] = R[[μ, ν, β, α]] = -R[[α, β, μ, ν]],
    {α, 2, n}, {β, α - 1}, {μ, 2, α}, {ν, If[μ == α, β, μ - 1]}];
    Do[R2[[μ, ν]] = Simplify[Sum[ing[[α, β]] * R[[α, μ, β, ν]], {α, n}, {β, n}]];  
    If[μ ≠ ν, R2[[ν, μ]] = R2[[μ, ν]], {μ, n}, {ν, μ}];
    R0 = Simplify[Sum[ing[[μ, ν]] R2[[μ, ν]] If[μ ≠ ν, 2, 1], {μ, n}, {ν, μ}]];  
    Do[If[R[[α, β, μ, ν]] != 0, Print["\"R\"Grid[""\"(\"α\"-1,\"β\"-1,\"μ\"-1,\"ν\"-1)\"\", R[[α, β, μ, ν]]],
          {α, 2, n}, {β, α - 1}, {μ, 2, α}, {ν, If[μ == α, β, μ - 1]}];
    Do[If[R2[[μ, ν]] != 0, Print["\"R\"Grid[""\"(\"μ\"-1,\"ν\"-1)\"\", R2[[μ, ν]]], {μ, n}, {ν, μ}];
    If[R0 != 0, Print["\"R \"", R0]];  
    {R, R2, R0}]
```

Here is the output for the Schwarzschild metric, assuming $r_S = 1$:

```math
\text{Riemann}[\text{Metric}[\{(1 - 1/r), (0, -1/(1 - 1/r)), (0, 0, -r^2), (0, 0, 0, -r^2 \sin[\theta]^2)\}]];
```

\[
\begin{align*}
\Gamma^0_{00} &= \frac{1}{2 r^2}, \\
\Gamma^0_{00} &= 0, \\
\Gamma^0_{11} &= \frac{1}{2 (-1 + r)^2}, \\
\Gamma^1_{22} &= 0, \\
\Gamma^1_{13} &= r \sin[\theta]^2, \\
\Gamma^2_{21} &= -r, \\
\Gamma^2_{33} &= r^2 \cos[\theta] \sin[\theta], \\
\Gamma^3_{31} &= -r \sin[\theta]^2, \\
\Gamma^3_{32} &= -r^2 \cos[\theta] \sin[\theta], \\
\Gamma^0_{10} &= \frac{1}{2 (-1 + r)^2 - (1 + r) r}, \\
\Gamma^1_{00} &= \frac{-1 + r}{2 r^2}, \\
\Gamma^1_{01} &= \frac{1}{2 r - 2 r^2}, \\
\Gamma^1_{22} &= 1 - r, \\
\Gamma^1_{33} &= -(1 + r) \sin[\theta]^2, \\
\Gamma^2_{21} &= \frac{1}{r}, \\
\Gamma^2_{33} &= -r \cos[\theta] \sin[\theta], \\
\Gamma^3_{31} &= 1, \\
\Gamma^3_{32} &= -r \cos[\theta], \\
R^1_{10} &= \frac{1}{r^3}, \\
R^1_{02} &= \frac{-1 + r}{2 r^2}, \\
R^1_{21} &= \frac{1}{2 (-1 + r)}, \\
R^3_{00} &= \frac{(-1 + r) \sin[\theta]^2}{2 r^2}, \\
R^3_{13} &= \frac{\sin[\theta]^2}{r}, \\
R^3_{11} &= -2 + 2 r, \\
R^3_{22} &= -r \sin[\theta]^2.
\end{align*}
\]
Thank You!

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