Coherent Follow-up of Continuous Gravitational Wave Candidates

M. Shaltev, R. Prior
Albert Einstein Institute, Hannover, Germany
miroslav.shaltev@aei.mpg.de, reinhard.prior@aei.mpg.de

LIGO-G1001169-v1

Direct detection of continuous emission of gravitational waves from spinning neutron stars is expected to become an important tool for neutron-star astrophysics. The weakness of the likely signals compared to the noise level of present instruments requires very long observation times. As a consequence, searches for previously unknown continuous gravitational wave sources must cover an enormous parameter space. Searching with fully coherent matched-filtering [1] is computationally prohibitive, and therefore advanced semi-coherent techniques [2, 3, 4, 5], such as those developed for the distributed computing project Einstein@Home [6], are used. Candidates from these searches require follow-up in only a small region of parameter space. Here we discuss two different methods to determine the required integration time \( T \) and related computing cost for a fully coherent follow-up of such candidates using the \( F \)-statistic [1].

Method 1: Expectations

We need to integrate long enough, such that the expected \( \bar{S}^2 \) value for a real signal is higher than the expected \( \bar{S}^2 \) value for Gaussian noise.

- **Expected value for real signal in fully-coherent \( F \)-statistic search:**
  \[
  E[\bar{S}^2] = 1 + \rho^2
  \]
  where \( \rho \) is signal to noise ratio (SNR).

- For an Einstein@Home candidate
  \[
  \rho^2 = \frac{N_S}{N_F} \frac{\bar{T}}{\bar{T}_F}
  \]
  where \( \rho_{\text{cand}}^2 = 2 \bar{T}_\text{cand} - 4 \) with \( 2 \bar{T}_\text{cand} \) the average \( S \) value of the segments, \( N_S \) number of detectors, \( N_F \) number of detectors used to select the candidate and \( \bar{T}_F \) segment duration.

- Probability to get maximal \( S \) value in \( N \) trials in Gaussian noise
  \[
  p_G = \left( \frac{N}{\sqrt{2\pi}} \right) \frac{1}{\rho_{\text{cand}}} \Gamma \left( \frac{N}{\rho_{\text{cand}}} \right)
  \]
  with \( \rho_{\text{cand}} \) central \( \chi^2 \) distribution, \( p_{\text{cand}} = 1 - \alpha \) and \( \alpha \) single trial false alarm.

- Taking into account signal \( \sigma(\bar{S}^2) \), and noise \( \sigma(\bar{S}^2) \) standard deviations, we require
  \[
  \frac{\bar{\bar{S}}}{\sigma(\bar{S}^2)} = n \frac{\bar{\bar{S}}}{\bar{S}^2} > E(\bar{S}^2) = \frac{\bar{T}}{\bar{T}_F}
  \]
  where \( n \) is a safety factor.

- For detection probability (DP) \( \eta = 0.9 \) by using Chebyshev’s inequality \( n \approx 4.11 \).

An illustrative example is given in Figure 1.

![Figure 1: Accumulated \( \bar{S}^2 \) value and standard deviation as function of \( T \) for a candidate with \( 2 \bar{T}_\text{cand} = 5.3 \) and expected \( \bar{S}^2 \) value and standard deviation for Gaussian noise. The cyan shaded area is the \( 2\sigma(\bar{S}^2) \) band around \( 1.4\bar{T} \). The red shaded area is the \( 2\sigma(\bar{S}^2) \) band around \( E(\bar{S}^2) \). The crossing point of the cyan and red curve is the required minimal coherent integration time.](image)

Method 2: Probabilities

We compute the required coherent integration time at set false alarm (FA) and false dismissal (FD) probabilities.

- Single trial FA probability
  \[
  \alpha_1 = (1 + F_{\text{cand}}) \exp(-F_{\text{cand}})
  \]

- Numerical solution yields threshold \( F_{\text{th}} \).

- For large number of templates \( N \) and overall FA probability \( \alpha \), we choose \( \alpha = \alpha_1 / N(T_\alpha) \).

- The TD probability and thus DP \( \eta = 1 - \beta \) is
  \[
  \beta(2\bar{F}_\text{tim}, \rho^2) = \int_{\rho^2}^{\infty} (2\bar{T}) pdf(2\bar{F}, \rho^2)
  \]

- Numerical solution yields threshold SNR \( \rho^2_{\text{th}} = \rho_{\text{cand}}^2 (\text{at chosen FA}) \) and FD \( \rho^2_{\text{FD}} \).

- For an Einstein@Home candidate
  \[
  \bar{T} = \frac{1}{N_{\text{cand}}} \frac{N_S}{N_F} \bar{T}_F
  \]

Comparison

To compare both methods we evaluate Eqs. 1 and 2 for a candidate with \( 2 \bar{T}_\text{cand} = [5 \ldots 6] \) and spin-down age \( \alpha = 2200 \text{yr} \) at frequency \( f = 185.0 \text{Hz} \) in a search volume defined by the Fisher matrix of trigger from the GC1 Einstein@Home run. The required integration time, number of templates and computing cost are plotted in Figures 2.3 and 4.

![Figure 2: Coherent integration time \( T \) in days as function of the strength of a candidate measured by \( 2\bar{T}_\text{cand} \). The blue curve is the required time computed by using expected values (method 1). The green curve is the time computed by using probabilities (method 2).](image)

![Figure 3: Required number of templates \( N \) as function of the strength of a candidate measured by \( 2\bar{T}_\text{cand} \). In accordance with the longer integration time of the first method, we need less templates when using the second method.](image)

![Figure 4: Computing cost \( C \) in hours as function of the strength of a candidate measured by \( 2\bar{T}_\text{cand} \) for recent hardware and version of the search code. The cost of the second method is significantly reduced due to the fewer number of templates.](image)

Monte Carlo Study

We use standard LIGO software (LALapps) to simulate signals in Gaussian noise and real data and measure the \( \bar{S}^2 \) values for integration times computed with the methods described above.

- **Method 1:** the required DP of 99% was exceeded at \( \bar{T} = 1850 \text{yr} \) for the simulated data and at \( \bar{T} = 990 \text{yr} \) for the injections in real data.

<table>
<thead>
<tr>
<th>Noise Trials</th>
<th>( \eta )</th>
<th>( n )</th>
<th>( 2\bar{T}<em>\text{cand} ) &gt; ( 2\bar{T}</em>\text{th} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>0.90</td>
<td>1.00</td>
<td>0.99</td>
</tr>
<tr>
<td>Ssa-H1</td>
<td>1.00</td>
<td>0.90</td>
<td>0.848</td>
</tr>
</tbody>
</table>

| Table 1: MC using expectations to compute \( T \) |

- **Method 2:** we required DP of 99%. We reached 99.7% DP for Gaussian noise and 94.4% for injections in real Ssa-H1 data.

<table>
<thead>
<tr>
<th>Noise Trials</th>
<th>( \eta )</th>
<th>( n )</th>
<th>( 2\bar{T}<em>\text{cand} ) &gt; ( 2\bar{T}</em>\text{th} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>0.90</td>
<td>1.00</td>
<td>0.897</td>
</tr>
<tr>
<td>Ssa-H1</td>
<td>0.90</td>
<td>0.897</td>
<td>0.848</td>
</tr>
</tbody>
</table>

| Table 2: MC using probabilities to compute \( T \) |

Conclusion

Method 2 of computing the required coherent integration time by using FA and FD probabilities is found to be preferable. The main reason for the discrepancy in \( \bar{T} \) for the two methods is the conservative estimation of the safety factor \( n \) in method 1, due to Chebyshev’s inequality being a lower bound. For real follow-ups we further need to extend method 2 to take into account noise-level changes and gaps in the data.

References