Signal detection is commonly based on generalized Neyman-Pearson test statistics, as in the case of the matched filter. In case of a non-detection, the detection statistic may be used to derive a loudest-event upper limit on the signal magnitude. Alternatively, within the Bayesian framework, the signal amplitude may be bounded via a posterior upper limit. We use a toy example to investigate the differences between the two kinds of limits.

Example problem
For illustration, we consider the example of detection of a sinusoidal signal

\[ s(t) = A \sin(2\pi f t + \phi) \]

in white Gaussian noise. This example shares many similarities with commonly encountered problems, e.g., nuisance parameters, partly analytical, partly numerical likelihood maximization,...

ML detection
In case of the sinusoidal signal model, the maximum likelihood detection statistic (matched filter) is the maximum of the periodogram,

\[ d^2 = \frac{2}{N} \sum_j |\hat{y}_j|^2 \]

where \( \hat{y} \) is the DFT of the data \( y \). Under the null hypothesis \( H_0 \) of “noise only”, the (normalized) power in all frequency bins \( \frac{2}{N} |\hat{y}_j|^2 \) follows a \( \chi^2 \) distribution. Under the alternative signal hypothesis \( H_1 \), the power in the (single) “true” frequency bin is noncentral-\( \chi^2 \)-distributed with noncentrality parameter \( \lambda = \frac{N}{2} \sigma^2 A^2 \) (“SNR”).

The distribution of the maximum \( d^2 \) is illustrated below: it depends on signal amplitude \( A \) and the trials factor \( m \), the number of Fourier frequency bins considered.

The loudest event upper limit
Detection significance and upper limit determination are based on the detection statistic’s distribution under \( H_0 \) and \( H_1 \), as shown below.

Example of detection and 90% upper limit construction (here: detection statistic \( d^2 = 11 \)).

Interpretation of the upper limit then is “if the amplitude been \( A^* \) or larger, a larger detection statistic value would have been observed with 90% probability.”

The posterior upper limit
A Monte Carlo study shows that the posterior limit is essentially identical when based on the complete data \( y \) or loudest event \( d^2 \) only. Assuming a uniform prior, the “loudest event” posterior limit is then computed based on integration over the likelihood function as shown below.

Integrations for both kinds of limit are very different.

Interpretation of the upper limit then is “if there is a signal, then its amplitude is less than \( A^* \) with 90% probability \( P(A < A^*|y, H_1) = 90% \).”

Comparison

The mapping from detection statistic to limit.

The 90% upper limits’ behaviour for different amplitudes.

The 90% upper limits’ behaviour under \( H_0 \) for increasing trials factor.

Under \( H_0 \), the posterior limit is more sensitive, especially for large trials factors.

Different 90% sensitivity limits.

The closely related sensitivity statement is actually very different, as it is inseparably connected to the detection problem, based on some false alarm probability, and is independent of the observed detection statistic value.